With this expression for $\gamma_2(t)$, Eq. (26) can be integrated. The Lagrangian multiplier consequently can be solved for as

$$\lambda = \frac{r_0^2 \dot{\theta}_0 (K+3)}{t_f^{K+3}} \tag{28}$$

With Eq. (28), we can solve for $u(t)$ from Eq. (26) as

$$u(t) = u(0) + \frac{\dot{\theta}_0}{t_f^{K+1}}\left[(t_f - t)^{K+3} - t_f^{K+3}\right] \tag{29}$$

The control acceleration $a_{M_\theta}(t)$ and the line-of-sight rate $\dot{\theta}(t)$ can be obtained as explicit functions of time as

$$a_{M_\theta}(t) = (K+3)\dot{r}_0\dot{\theta}_0[1 - (t/t_f)]^{K+1}, \qquad K \neq -3 \tag{30}$$

and

$$\dot{\theta}(t) = \dot{\theta}_0[1 - (t/t_f)]^{K+1} \tag{31}$$

By varying K ($\neq -3$), we can obtain a family of proportional navigation guidance laws. In particular, let $K = 0$ in Eqs. (30) and (31). We get

$$a_{M_\theta}(t) = 3\dot{r}_0\dot{\theta}_0[1 - (t/t_f)] \tag{32}$$

and

$$\dot{\theta}(t) = \dot{\theta}_0[1 - (t/t_f)] \tag{33}$$

The plots of nondimensionalized acceleration variations and nondimensionalized line-of-sight rate variations with nondimensionalized time for different K are presented in Figs. 2 and 3, respectively. It can be seen that, as K increases, the initial line-of-sight rates are brought down quickly with higher levels of acceleration. In contrast, intermediate values of acceleration allow initial lines-of-sight to remain fairly high (good for the estimator) and then drive the line-of-rate to zero to effect the intercept. If K is low ($n = 0$), higher levels of accelerations are required, even near the end.

IV. Conclusions

A class of proportional navigation guidance laws has been derived through an approximation of time-to-go and a transformation of state variables.

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Linear Quadratic Pursuit–Evasion Games with Terminal Velocity Constraints

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I. Introduction

MODERN guidance techniques are based mainly on optimal control theory and on differential games. Reference 1 compares these methods and advocates the use of differential game guidance methods, particularly against maneuvering targets. The widely used proportional navigation is in itself an optimal strategy in which the cost is the missile's control effort and a zero miss (against a nonmaneuvering target) is imposed by the boundary conditions. In optimal rendezvous[2] the same cost is employed with the additional constraint of zero terminal lateral velocity (with respect to a nominal line-of-sight). For some cases, this additional requirement may be of importance from an operational or technical point of view. In a well-known differential game (originally solved in Ref. 3) the target is maneuvering to maximize a weighted sum of miss distance, the missile's control effort, and its own control effort (the latter with negative weight). The pursuer is trying to minimize the same cost function (a zero-sum game). The strategy obtained by this classical work will be referred to as the game-theoretic optimal intercept.

The present work is a natural extension of the last two cases. The differential-game approach is used where the terminal lateral velocity is introduced into the cost in addition to the adversaries control efforts and the miss distance. The case of zero terminal conditions will naturally be called the game-theoretic optimal rendezvous

This Note is organized as follows. After introducing the governing equations, the general problem will be formulated. The problem will be solved in closed form and the above-mentioned cases of one-sided optimal rendezvous and game-theoretic optimal intercept will be shown to be special cases of this problem. We will then concentrate on perfect (zero-miss) intercepts with terminal velocity constraints and, in particular, on the game-theoretic optimal rendezvous for which optimal strategies as well as the resulting optimal trajectories will be given in closed form.

II. Problem Formulation

We make the following assumptions:

1) The pursuit–evasion conflict is two dimensional, in the horizontal plane.

2) The speeds of the pursuer P and the evader E are constant.

3) The trajectories of P and E can be linearized around their collision course.

4) Both opponents can directly control their lateral accelerations.

5) The pursuer is more maneuverable than the evader.

Under these assumptions the problem has the following state-space representation[2]:

$$\dot{X} = AX + Bu + Dw$$
$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad D = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{1}$$

where x_1 and x_2, the components of X, are the relative displacement and velocity and u and w are the normal accelerations of the pursuer and the evader, respectively.

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The minimax problem is given as

$$\min_u \max_w J = \frac{b}{2}x_1^2(t_f)+\frac{c}{2}x_2^2(t_f)+\frac{1}{2}\int_0^{t_1} u^2(t)-\gamma^2 w^2(t)\,dt \quad (2)$$

where $b > 0$ and $\gamma > 1$. The latter condition implies that the evader is less maneuverable than the pursuer; hence the penalty on its control effort is higher. Notice that c is not assumed to be necessarily positive; hence we consider cases where the pursuer's objective is to minimize the terminal lateral velocity ($c > 0$), cases where this variable is of no importance ($c = 0$), and cases where the pursuer's objective is to maximize this velocity ($c < 0$). This objective depends heavily on the particular conflict for which the problem is formulated.

Perfect intercept is modeled with $b \to \infty$, and finally for game-theoretic optimal rendezvous we have $b \to \infty$, $c \to \infty$.

III. Optimal Feedback Solution

The Riccati equation for the general problem is the following[4]:

$$-\dot{P} = PA + A^T P - PBB^T P + \gamma^{-2} PDD^T P \quad (3)$$

where the terminal conditions are

$$P(t_l) = \begin{bmatrix} b & 0 \\ 0 & c \end{bmatrix}$$

and the optimal strategies are of the form

$$\begin{aligned} u &= -B^T PX = -g_1 x_1 - g_2 x_2 \\ w &= D^T \gamma^{-2} PX = \gamma^{-2}(g_1 x_1 + g_2 x_2) \end{aligned} \quad (4)$$

We can formulate the feedback gains in closed form as follows (where $\tau = t_f - t$):

$$\begin{aligned} g_1 &= \frac{b\gamma^2\tau\left[\frac{1}{2}\tau c(\gamma^2-1)+\gamma^2\right]}{\tau^4\left(\frac{1}{12}cb\right)(\gamma^2-1)^2+\left[\tau c+\tau^3\left(\frac{1}{3}b\right)\right]\gamma^2(\gamma^2-1)+\gamma^4} \\ g_2 &= \frac{\frac{1}{3}\gamma^2\left[\tau^3 cb(\gamma^2-1)+3(\tau^2 b+c)\gamma^2\right]}{\tau^4\left(\frac{1}{12}cb\right)(\gamma^2-1)^2+\left[\tau c+\tau^3\left(\frac{1}{3}b\right)\right]\gamma^2(\gamma^2-1)+\gamma^4} \end{aligned} \quad (5)$$

Notice, however, that even though $\gamma > 1$ (which is a sufficient condition for the existence of a solution in the infinite-horizon case), the solution here is not guaranteed for $c < 0$! We shall come back to this point later, but first we will present two known cases that can be derived from the general solution.

IV. Two (Previously Known) Special Cases

A. Game-Theoretic Optimal Intercept

If in Eq. (5) we let $c = 0$, then

$$g_1 = \frac{\tau}{\frac{1}{3}\tau^3(1-\gamma^{-2})+1/b} \qquad g_2 = \frac{\tau^2}{\frac{1}{3}\tau^3(1-\gamma^{-2})+1/b} \quad (6)$$

which are known results (see Ref. 2, p. 287).

Additionally, taking $b \to \infty$, we get

$$g_1 = \frac{3}{\tau^2(1-\gamma^{-2})} \qquad g_2 = \frac{3}{\tau(1-\gamma^{-2})} \quad (7)$$

which are the feedback gains for the game-theoretic optimal intercept.

Finally, in this special case, as $\gamma \to \infty$, proportional navigation gains are obtained.

B. One-Sided Optimal Rendezvous

If in Eq. (5) we let $\gamma \to \infty$, then

$$\begin{aligned} g_1 &= \frac{b\tau\left(\frac{1}{2}\tau c+1\right)}{\tau^4\left(\frac{1}{12}cb\right)+\tau c+\tau^3\left(\frac{1}{3}b\right)+1} \\ g_2 &= \frac{\frac{1}{3}\tau^3 cb+\tau^2 b+c}{\tau^4\left(\frac{1}{12}cb\right)+\tau c+\tau^3\left(\frac{1}{3}b\right)+1} \end{aligned} \quad (8)$$

These are also known results (see Ref. 2, p. 154).

Taking $b \to \infty$, $c \to \infty$, we get

$$g_1 = 6/\tau^2 \qquad g_2 = 4/\tau \quad (9)$$

which are the coefficients for the one-sided optimal perfect rendezvous.[2]

V. Perfect Intercept with Terminal Velocity Constraints

The main results of this Note are obtained by letting $b \to \infty$ in Eq. (5) [i.e., $X_1(t_f) = 0$]. We get

$$\begin{aligned} g_1 &= \frac{6\tau + 12/c(1-\gamma^{-2})}{\tau^3(1-\gamma^{-2})+4\tau^2/c} \\ g_2 &= \frac{4\tau + 12/c(1-\gamma^2)}{\tau^2(1-\gamma^{-2})+4\tau/c} \end{aligned} \quad (10)$$

For the case $c > 0$, the solution exists for all $\tau \in (0, t_f)$. For $c < 0$ the solution exists only if

$$\tau < -\frac{4}{c(1-\gamma^{-2})}$$

Since the maximum value on the left-hand side is t_f, we obtain the condition

$$t_f < -\frac{4}{c(1-\gamma^{-2})} \quad (11)$$

It has been shown[5] that for $\gamma \to \infty$ (the one-sided optimal control problem) the last inequality is the Jacobi condition for no conjugate points on the interval $(0, t_f)$.

The perfect game-theoretic rendezvous is obtained directly from Eq. (5) by letting $b \to \infty$, $c \to \infty$. We obtain the feedback gains

$$g_1 = \frac{6}{\tau^2(1-\gamma^{-2})} \qquad g_2 = \frac{4}{\tau(1-\gamma^{-2})} \quad (12)$$

This very simple form constitutes a natural extension of Eqs. (7) and (9). The corresponding equations of motion (assuming both players are playing optimally) become

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u + w = -(1-\gamma^{-2})(g_1 x_1 + g_2 x_2) \\ &= (-6/\tau^2)x_1 + (-4/\tau)x_2 \end{aligned} \quad (13)$$

Notice the interesting result that the governing equations are identical to those of the one-sided optimal rendezvous with a nonmaneuvering target and are independent of γ! (A similar result holds in the game-theoretic optimal-intercept case.)

For these simple equations there exists a closed-form expression for the trajectories

$$x_1 = c_1\tau^2 + c_2\tau^3 \qquad x_2 = -2c_1\tau - 3c_2\tau^2 \quad (14)$$

where the constant c_1, c_2 can be determined from the initial conditions as

$$\begin{aligned} c_1 &= (3/t_f^2)x_1(0) + 1/(t_f)x_2(0) \\ c_2 &= (-2/t_f^3)x_1(0) - (1/t_f^2)x_2(0) \end{aligned} \quad (15)$$

VI. Conclusions

A large family of feedback solutions has been obtained from a simple-looking performance index that contains a weighted combination of the pursuer's control effort, the evader's control effort, the miss distance, and the terminal lateral velocity. Of particular interest are trajectories that terminate with capture (perfect intercepts) but with various values for the terminal lateral velocity. A perfect rendezvous results in solutions to the optimization problem that are simple and may be easily expressed in closed form.

The models used in this research are extremely simplified. Further investigation should be carried out on more general models with realistic dynamics.

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Dense–Sparse Discretization for Optimization and Real-Time Guidance

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Introduction

A COMMON practice for onboard implementation of optimal or near-optimal guidance strategies is to solve the trajectory optimization problem offline for various boundary conditions. Numerous reference trajectories and the associated gains are then evaluated based on neighboring optimal control concepts[1] and are stored onboard. These reference trajectories and gains are used onboard with state and target observation or estimation to quickly evaluate control actions via simple linear feedback laws. However, this requires extensive ground-based analysis and onboard storage capacity.

For rapid trajectory prototyping, the safest and most robust approaches are the direct methods. These methods rely on a finite dimensional discretization of the optimal control problem to nonlinear programming problems. Even though these methods do not enjoy the high precision and resolution of indirect methods, their convergence robustness makes them the method of choice for most practical applications. Moreover, these methods do not require the advanced mathematical skills required to pose and solve the variational problem. With the advent of high-power computing and excellent nonlinear programming tools, direct methods have been used extensively to calculate optimal trajectories offline. Some of the most common approaches include control discretization, as in the program to optimize simulated trajectories (POST) software,[2] collocation-based methods[3] as in the optimal trajectories by implicit simulation (OTIS) code, and a recently introduced trajectory optimization via differential inclusions (TODI)[4] method.

Although all of the cited direct approaches have been used for offline optimization of trajectories, real-time online guidance strategies using these methods have not yet been developed and implemented. This Note proposes such a guidance scheme. In each control evaluation step a rough near-optimal solution to the current optimal control problem is generated using an intuitive dense–sparse discretization. The direct method employed throughout this paper is the differential inclusion approach in conjunction with NPSOL[5] as the optimization engine, although any other robust optimization technique could be utilized. A minimum time-to-climb problem[6] of an F-15 aircraft is used as an example to illustrate the concept.

Dense–Sparse Discretization

If one is driving from New York to Los Angeles, and the objective is to minimize fuel or time, would it really be necessary to make instantaneous control actions to account for every curve and pothole miles away? How much does the performance index improve by doing this? Although the optimal solution is influenced by every curve and pothole miles away, intuitively it is clear that the degradation in performance by neglecting these would be only marginal.

By experience, TODI captures the general profile of the optimal solution well even with a very small number of nodes. (The larger the number of nodes, the more precise the solution, but the CPU time and memory required to solve the nonlinear programming problem goes up nonlinearly, i.e., approximately as the third power of the number of parameters for a nonsparse solver and as the second power for a sparse solver.) Hence, it is proposed to discretize the trajectory with nodes densely placed near the current time to capture immediate dynamics well and with nodes sparsely placed for the rest of the trajectory to approximate the overall trend. This simple idea is used to determine the near-optimal control action at the current time. Clearly, a whole range of alternate methods to achieve the desired high resolution near initial time can be conceived, but for the ease of presentation, the remainder of this Note is restricted exclusively to the dense–sparse discretization just introduced. The algorithm is discussed in the following section.

Let the speed and computing power available limit the discretization for the problem at hand to, say, eight nodes. The speed and CPU limit may be the limits on a workstation/personal computer performing offline optimization or it may correspond to the limitations of an onboard guidance computer. Place N_D dense nodes (say, four) close to initial time and place N_S sparse nodes (four) for the rest of the trajectory. The nodal density may be chosen by the user. The resulting nonlinear programming problem with eight nodes is then solved. This concludes step 1 of the algorithm as shown in Fig. 1.

During step 2 of the optimization scheme, a displacement integer m is defined that specifies the number of nodes to accept as optimal.

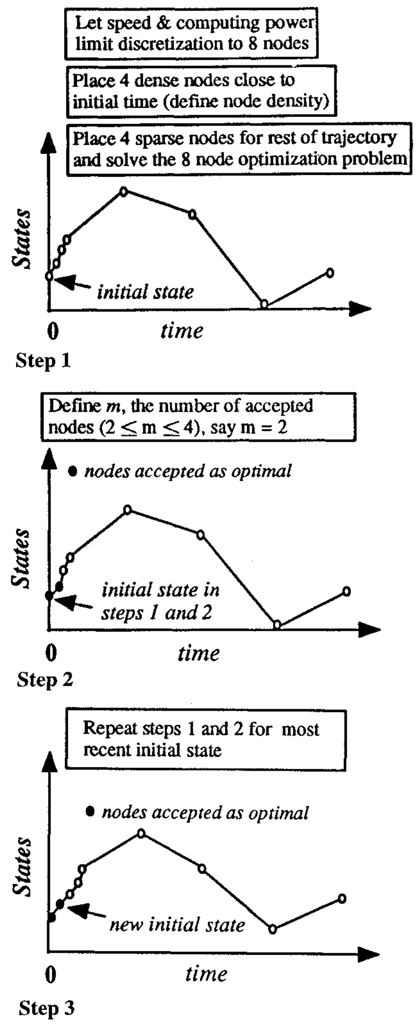

Fig. 1 Steps of optimization/guidance algorithm.

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